Exercise 10

As in Exercise 9, consider a spring with mass m, spring constant k, and damping constant $c = 0$, and let $\omega = \sqrt{k/m}$. If an external force $F(t) = F_0 \cos \omega t$ is applied (the applied frequency equals the natural frequency), use the method of undetermined coefficients to show that the motion of the mass is given by

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t + \frac{F_0}{2m\omega} t \sin \omega t$$

Solution

To obtain the equation of motion for a mass attached to a spring that experiences damping and has an external force $F(t)$ applied to it, use Newton's second law.

$$\sum F = ma$$

$$F(t) - c\frac{dx}{dt} - kx = m\frac{d^2x}{dt^2}$$

Solve for $F(t) = F_0 \cos \omega t$ and set $c = 0$, since the damping is negligible.

$$m\frac{d^2x}{dt^2} + kx = F_0 \cos \omega t$$

Divide both sides by m.

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = \frac{F_0}{m} \cos \omega t$$

Let $\omega^2 = k/m$.

$$\frac{d^2x}{dt^2} + \omega^2 x = \frac{F_0}{m} \cos \omega t \tag{1}$$

Since this ODE is linear, the general solution can be expressed as the sum of a complementary solution and a particular solution.

$$x = x_c + x_p$$

The complementary solution satisfies the associated homogeneous equation.

$$\frac{d^2x_c}{dt^2} + \omega^2 x_c = 0 \tag{2}$$

This is a linear homogeneous ODE, so its solutions are of the form $x = e^{rt}$.

$$x_c = e^{rt} \quad \rightarrow \quad \frac{dx_c}{dt} = re^{rt} \quad \rightarrow \quad \frac{d^2x_c}{dt^2} = r^2 e^{rt}$$

Plug these formulas into equation (1).

$$r^2 e^{rt} + \omega^2 (e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + \omega^2 = 0$$

Solve for r.

$$r = \{-i\omega, i\omega\}$$

Two solutions to equation (2) are $e^{-i\omega t}$ and $e^{i\omega t}$. By the principle of superposition, then,

$$\begin{aligned}x_c(t) &= C_1e^{-i\omega t} + C_2e^{i\omega t} \\ &= C_1(\cos\omega t - i\sin\omega t) + C_2(\cos\omega t + i\sin\omega t) \\ &= (C_1 + C_2)\cos\omega t + (-iC_1 + iC_2)\sin\omega t \\ &= C_3\cos\omega t + C_4\sin\omega t,\end{aligned}$$

where C_3 and C_4 are arbitrary constants. On the other hand, the particular solution satisfies the original ODE.

$$\frac{d^2x_p}{dt^2} + \omega^2 x_p = \frac{F_0}{m}\cos\omega t \tag{3}$$

Since the inhomogeneous term is a cosine function, the particular solution would be $x_p = A\cos\omega t + B\sin\omega t$. $\cos\omega t$ satisfies the homogeneous equation, though, so an extra factor of t is needed: $x_p = t(A\cos\omega t + B\sin\omega t)$.

$$\begin{aligned}x_p &= t(A\cos\omega t + B\sin\omega t) \\ \frac{dx_p}{dt} &= (A\cos\omega t + B\sin\omega t) + t(-A\omega\sin\omega t + B\omega\cos\omega t) \\ \frac{d^2x_p}{dt^2} &= (-A\omega\sin\omega t + B\omega\cos\omega t) + (-A\omega\sin\omega t + B\omega\cos\omega t) + t(-A\omega^2\cos\omega t - B\omega^2\sin\omega t)\end{aligned}$$

Substitute these formulas into equation (3).

$$[(-A\omega\sin\omega t + B\omega\cos\omega t) + (-A\omega\sin\omega t + B\omega\cos\omega t) + t(-A\omega^2\cos\omega t - B\omega^2\sin\omega t)] + \omega^2[t(A\cos\omega t + B\sin\omega t)] = \frac{F_0}{m}\cos\omega t$$

Simplify the left side.

$$2\omega B\cos\omega t - 2\omega A\sin\omega t = \frac{F_0}{m}\cos\omega_0 t$$

Match the coefficients to get a system of equations involving A and B.

$$\begin{aligned}2\omega B &= \frac{F_0}{m} \\ -2\omega A &= 0\end{aligned}$$

Solving it yields $A = 0$ and $B = F_0/(2m\omega)$. Therefore, the particular solution is

$$x_p = \frac{F_0}{2m\omega}t\sin\omega t,$$

and the general solution to equation (1) is

$$\begin{aligned}x(t) &= x_c + x_p \\ &= C_3\cos\omega t + C_4\sin\omega t + \frac{F_0}{2m\omega}t\sin\omega t.\end{aligned}$$